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# Modulating properties of harmonic breather solutions of KdV

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**Abstract.** We show that a class of solutions of KdV can be used to modulate miscellaneous wavepackets similarly to how the trigonometric functions are used in linear theory to perform the same task.

The theory of Fourier transform is based on viewing miscellaneous functions $f(x,t)$ as linear superpositions $\int_{\mathbb{R}}[f_s(\lambda)\sin(\lambda x-\omega(\lambda)t)+f_c(\lambda)\cos(\lambda x-\omega(\lambda)t)]\,\mathrm{d}\lambda$ of trigonometric functions $\sin(\lambda x-\omega(\lambda)t)$ and $\cos(\lambda x-\omega(\lambda)t)$. It is often said that the trigonometric functions linearly modulate $f(x,t)$.

Among other things linear modulation is being used to construct localized in-space solutions of linear partial differential equations, so-called wave-packets. Such solutions find applications in electrodynamics, fluids and especially in quantum mechanics, where the interplay between the concepts of a particle and a wave is of fundamental importance. Yet the construction of wavepackets is severely limited by the requirement of linearity of the corresponding partial differential equations. Should the equations become even mildly nonlinear we can no longer construct exact wavepacket solutions. In such cases the best that we can do is to construct wavepacket solutions of the linearized partial differential equations and try to add a perturbation to accommodate the nonlinearities. Besides the fact that a solution so obtained is only an *approximate* solution, the introduction of a perturbation may also destroy space localization of the solution. It is only natural to ask if there could exist a nonlinear analogue of linear modulation that would allow us to construct *exact* wavepacket solutions for at least some nonlinear partial differential equations. The analogy between the Fourier transform and the inverse scattering method when used as a tool for solving the Cauchy problem for integrable equations, suggests that if any nonlinear partial differential equations allow nonlinear modulation then the integrable equations are the most likely candidates. Indeed, as it turns out, even the simplest representative of the integrable equations, the KdV equation

$$u_t+6uu_x+u_{xxx}=0 \qquad \lim_{x\to\pm\infty}u(t,x)=0 \tag{1}$$

permits nonlinear modulation.

We start by looking at the spectral problem associated with a potential $u$

$$-\frac{\partial^2\psi}{\partial x^2}-u\psi=\lambda^2\psi \qquad \psi(x,k)\sim \mathrm{e}^{\mathrm{i}kx} \qquad x\to+\infty.$$

Let $a(k)$ and $b(k)$ be determined by the asymptotic behaviour of $\psi(x,k)$ as $x$ approaches $-\infty$

$$\psi(x,k)\sim a(k)\mathrm{e}^{\mathrm{i}kx}-b(-k)\mathrm{e}^{-\mathrm{i}kx} \qquad x\to-\infty.$$

  

and $r(k) = \frac{b(k)}{a(k)}$. The solution $u(t, x)$ of (1) is then obtained via $u(t, x) = \frac{\mathrm{d}}{\mathrm{d}x} K(x, x)$, where $K(x, y)$ is, in turn, the solution of an integral equation $K(x, y) + F(x + y) + \int_x^{+\infty} K(x, z)F(y + x)\,\mathrm{d}z = 0$ with $F(z) = \sum_{j=1}^{N} \frac{\mathrm{i}b(k_j)}{a'(k_j)} \mathrm{e}^{\mathrm{i}k_j z} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} r(k)\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k$, [6, 12].

If $r(k) \equiv 0$ we obtain an $N$-soliton solution which already represents a nonlinear superposition of $N$ solitons with each soliton considered as its elementary component. Although each soliton can be viewed as somewhat of a particle, we cannot consider it to be a wavepacket for it has no oscillations within it.

Let us now turn our attention to the solitonless case, i.e. the case when $F(z) = \int_{-\infty}^{+\infty} r(k)\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k$, $r(-k) = \bar{r}(k)$ is a superposition of linear harmonics $\mathrm{e}^{\mathrm{i}kz}$. The most elementary potentials are obtained when $F(z)$ contains only two harmonics $\mathrm{e}^{\mathrm{i}\lambda z}$ and $\mathrm{e}^{-\mathrm{i}\lambda z}$, or more precisely, when we choose

$$r(k) = \lim_{\varepsilon \to 0} \begin{cases} \mathrm{e}^{\mathrm{i}\gamma + p\varepsilon + 8\mathrm{i}(\lambda + \mathrm{i}\varepsilon)^3 t} & |k - \lambda| < \varepsilon \\ \mathrm{e}^{-\mathrm{i}\gamma + p\varepsilon + 8\mathrm{i}(-\lambda + \mathrm{i}\varepsilon)^3 t} & |k + \lambda| < \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

One can explicitly compute these elementary potentials by essentially repeating the steps in derivation of the two-soliton solutions and, just like with the two-soliton solutions, these potentials can be written up by taking a $2 \times 2$ matrix $A$ with entries

$$A_{nm} = \delta_{nm} + \frac{\beta_n \mathrm{e}^{8\mu_n^3 t - (\mu_n + \mu_m)x}}{\mu_n + \mu_m}$$

$$\mu_1 = \mathrm{i}\lambda + \varepsilon \qquad \mu_2 = -\mathrm{i}\lambda + \varepsilon \qquad \beta_1 = 2\varepsilon \mathrm{e}^{\mathrm{i}\gamma + 2p\varepsilon} \qquad \beta_2 = 2\varepsilon \mathrm{e}^{-\mathrm{i}\gamma + 2p\varepsilon}$$

and letting

$$u = 2 \lim_{\varepsilon \to 0} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det A = \frac{8\lambda^2 \sin(8\lambda^3 t + 2\lambda x - \gamma)}{\sin(8\lambda^3 t + 2\lambda x - \gamma) - 2\lambda(12\lambda^2 t + x - p)} + 8\lambda^2 \left[ \frac{1 - \cos(8\lambda^3 t + 2\lambda x - \gamma)}{\sin(8\lambda^3 t + 2\lambda x - \gamma) - 2\lambda(12\lambda^2 t + x - p)} \right]^2. \tag{2}$$

The graph of (2) is shown in figure 1. These solutions have been previously derived by a number of authors but it is only in the last several years that their properties have attracted more careful studies, for example [12–14, 17–19], where (2) are correspondingly referred to as harmonic breathers or positons. Each harmonic breather is determined by three constants $\lambda$, $p$ and $\gamma$ which we refer to correspondingly as the frequency, displacement and phase.

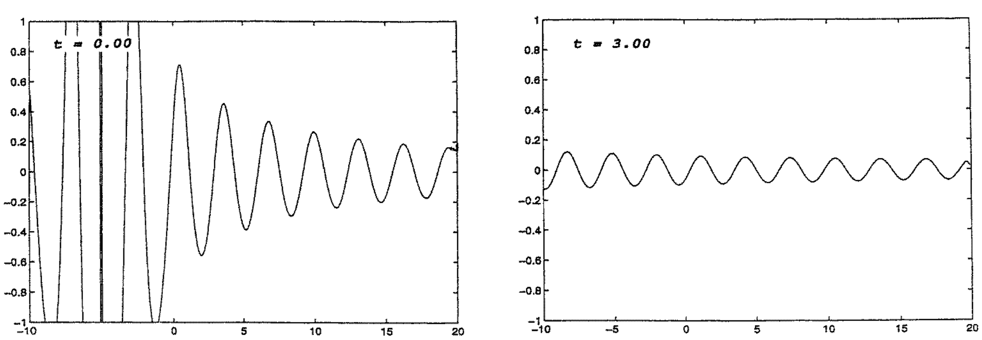


**Figure 1.** Snapshots of a single harmonic breather with $\gamma = 0$; $\lambda = 1$; $p = -5$ for the values of $t$ shown in the upper left corner.

By analogy with solitons we define the nonlinear superposition of $N$ harmonic breathers whose frequencies, displacements and phases are correspondingly given by sets $\Lambda = (\lambda_1, \ldots, \lambda_N)$, $P = (p_1, \ldots, p_N)$ and $\Gamma = (\gamma_1, \ldots, \gamma_N)$ to be the potential generated by taking

$$r(k) = \lim_{\varepsilon \to 0} \sum_{j=1}^{N} [\mathrm{e}^{\mathrm{i}\gamma_j + p_j\varepsilon} \chi_{[\lambda_j - \varepsilon, \lambda_j + \varepsilon]}(k) + \mathrm{e}^{-\mathrm{i}\gamma_j + p_j\varepsilon} \chi_{[-\lambda_j - \varepsilon, -\lambda_j + \varepsilon]}(k)]$$

where

$$\chi_{[\alpha,\beta]}(k) = \begin{cases} 1 & \alpha < k < \beta \\ 0 & \text{otherwise.} \end{cases}$$

Again, the corresponding potential can be computed in essentially the same manner as the $2N$-soliton solution [12, 13] yielding

$$u = 2 \lim_{\varepsilon \to 0} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det A \tag{3a}$$

where $A$ is a $2N \times 2N$ matrix with entries

$$A_{nm} = \delta_{nm} + \frac{\beta_n \mathrm{e}^{8\kappa_n^3 t - (\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m} \tag{3b}$$

$$\kappa_{2q-1} = \mathrm{i}\lambda_q + \varepsilon \qquad \kappa_{2q} = -\mathrm{i}\lambda_q + \varepsilon \qquad \beta_{2q-1} = 2\varepsilon \mathrm{e}^{\mathrm{i}\gamma_q + 2p_q\varepsilon} \qquad \beta_{2q} = 2\varepsilon \mathrm{e}^{-\mathrm{i}\gamma_q + 2p_q\varepsilon} \tag{3c}$$

$\lambda_q, \varepsilon > 0$ for $1 \leqslant q \leqslant N$.

Formulae (3) describe the motion and interaction of $N$ harmonic breathers, when $N = 1$ we recover (2). Each $N$-harmonic breather solution (3) is a meromorphic function with generally $N$ real poles, although as shown in [12] for some values of $t$ the number of real poles may be less than $N$. The behaviour of the poles and their interaction are described in [12, 13, 17]. Away from the poles the $N$-harmonic breather solutions are oscillatory, decaying to zero at infinity as $\mathrm{O}\left(\frac{1}{|x|}\right)$ as shown in [13]. The two-harmonic breather solution, which is obtained by taking $N = 2$ in (3), can be written as [11, 12]

$$u(t, x) = 2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ell\mathrm{n}(\tau_1 \tau_2 - q^2) \tag{4a}$$

where

$$\tau_k = p_k - 12\lambda_k^2 t - x - \frac{\sin(8\lambda_k^3 t + 2\lambda_k x - \gamma_k)}{2\lambda_k} \qquad k = 1, 2 \tag{4b}$$

and

$$q = \left[ \frac{\sin((4\lambda_1^3 t + \lambda_1 x - \gamma_1/2) - (4\lambda_2^3 t + \lambda_2 x - \gamma_2/2))}{\lambda_1 - \lambda_2} - \frac{\sin((4\lambda_1^3 t + \lambda_1 x - \gamma_1/2) + (4\lambda_2^3 t + \lambda_2 x - \gamma_2/2))}{\lambda_1 + \lambda_2} \right]. \tag{4c}$$

Formulae (4) generally make sense only when $\lambda_2 \neq \pm\lambda_1$ but for $\gamma_2 = \gamma_1 + 2n\pi$ the concept of the superposition of two harmonic breathers can be naturally extended to the case $\lambda_2 = \pm\lambda_1$ by taking the limit of (4) as $\lambda_2 \to \lambda_1$. The answer turns out to be a single harmonic breather solution with $\lambda = \lambda_1, \gamma = \gamma_1$ and $p = \frac{p_1 p_2}{p_1 + (-1)^n p_2}$, [11, 12].

Let us take a finite region $\Omega$ sufficiently far away from the poles of both harmonic breathers. For simplicity we can choose $\Omega = \{x, t||x| \leqslant X, 0 \leqslant t \leqslant T\}$ and $|p_1|, |p_2|, |p| \gg X, T$. Then in $\Omega$ the first and second harmonic breathers are equal to

$$\frac{4\lambda_k \sin(8\lambda_k^3 t + 2\lambda_k x - \gamma_k)}{p_k} + \mathrm{O}\left(\frac{1}{p_k}\right) \qquad k = 1, 2 \tag{4d}$$

i.e. up to the terms $\mathrm{O}(1/p_k)$ they are oscillatory waves of amplitude $1/p_k$.

The nonlinear harmonic breather obtained as the result of their superposition is equal in $\Omega$ to

$$\frac{4\lambda_1 \sin(8\lambda_1^3 t + 2\lambda_1 x - \gamma_1)}{p} + \mathrm{O}\left(\frac{1}{p}\right) \tag{4e}$$

i.e. the same oscillatory wave with amplitude $4\lambda/p = 4\lambda/p_1 + (-1)^n(4\lambda/p_2)$. Thus away from the poles a phenomenon closely resembling linear interference takes place, we call it nonlinear interference. Since it is exactly the phenomenon of linear interference of trigonometric functions that is responsible for the formation of wavepackets, the presence of its nonlinear analogue should also lead to the formation of wavepackets, this time, however, nonlinear. The simplest way of verifying whether it is actually so is to construct nonlinear wavepackets. Due to the lack of a better method, in our choice of parameters we will be guided by the analogy with the formula $\int_{\mathbb{R}}[f_s(\lambda)\sin(\lambda x - \omega(\lambda)t) + f_c(\lambda)\cos(\lambda x - \omega(\lambda)t)]\,\mathrm{d}\lambda$ for construction of linear wavepackets.

Since $\frac{4\lambda_k}{p_k}$ appears in (4*d*) and (4*e*) as an approximate amplitude of the oscillatory waves, we assume that it is this quantity that provides a measure of relative contribution of each wave in the wavepacket similar to the role played by $f_s(\lambda)$ and $f_c(\lambda)$ in the linear case.

Similar to the linear case, the simplest modulated wave is obtained by taking superposition $u(t, x)$ of two harmonic breathers with close values of $\lambda_1$ and $\lambda_2$. The graph of such $u(t, x)$ along with $(-p_1 + 12\lambda_{\mathrm{Ave}}^2 t + x)u(t, x)$, $\lambda_{\mathrm{Ave}} = (\lambda_1 + \lambda_2)/2$ is shown in figure 2. Although $u(t, x)$ decays to 0 as $|t|$ or $|x| \to \infty$, the graph of $(-p_{\mathrm{Ave}} + 12\lambda_{\mathrm{Ave}}^2 t + x)u(t, x)$ asymptotically becomes the same as a linearly modulated wavetrain as $|t|$ or $|x| \to +\infty$.

A more localized wavepacket is constructed by taking sets

$$\Lambda = (\lambda_1, \ldots, \lambda_N); \qquad \Gamma = (\gamma, \ldots, \gamma); \qquad P = (p_1, \ldots, p_N);$$
$$p_n = -p_0 \mathrm{e}^{a(\lambda_n - \lambda_{\mathrm{Ave}})^2}; \qquad p_0, a > 0.$$

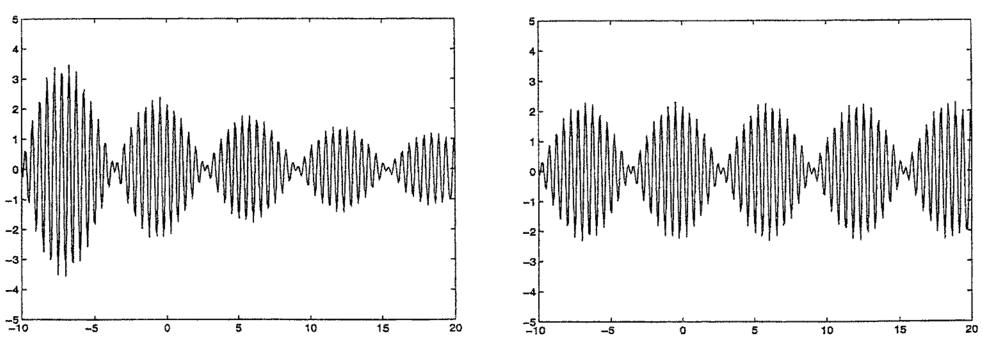


**Figure 2.** Snapshots of the wavetrain generated by two harmonic breathers with $\Gamma = (\pi/2, \pi/2\ )$; $\Lambda = (6, 6.25)$; $P = (-20, -20)$ (shown on the left-hand side) and of the same wavetrain multiplied by $(20 + 12\lambda_{\mathrm{ave}}^2 t + x)$, $\lambda_{\mathrm{ave}} = 6.25$ (shown on the right-hand side). For both $t = 0$.

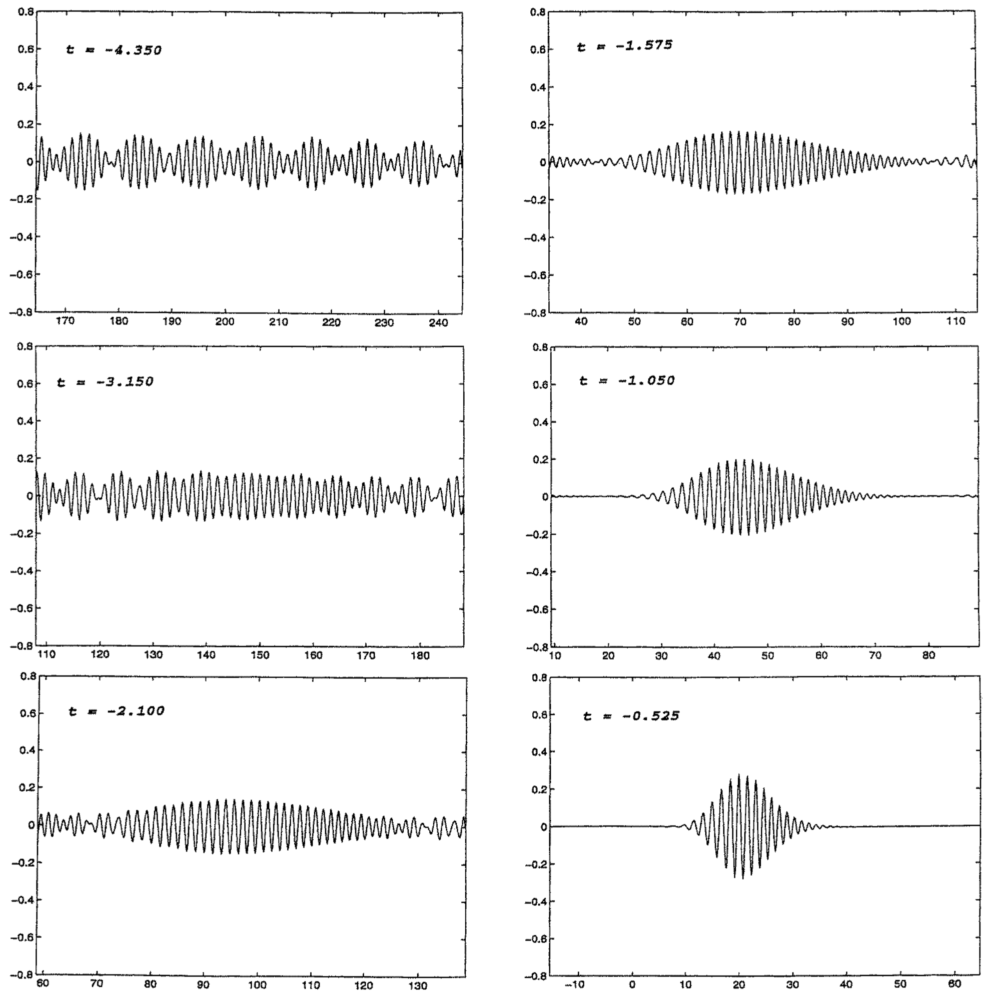


**Figure 3.** Formation and time evolution of the wavepacket generated by 21 harmonic breathers with $\Gamma = (0, 0, \ldots, 0)$; $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{20}, \lambda_{21})$, $\lambda_n = 1.5 + 0.05(n-1)$; $P = (p_1, \ldots, p_{21})$, $p_n = -100\lambda_n \mathrm{e}^{15}(\lambda_n - 2)^2$. The value of $t$, corresponding to each frame, is shown in the upper left corner of each frame.

Evolution of such wavepackets is shown in figure 3, and as we can see the wavepacket is localized within the set $\Delta = \{t, x \| t| < T, |x - 12\lambda_{\mathrm{Ave}}^2 t| < X\}$. In the example shown in figure 3, constants $T = 1$, $X = 40$, $\lambda_{\mathrm{Ave}} = 2$. As $t \to +\infty$ the wavepacket propagates to the left and eventually disappears due to dispersion of the harmonic breathers compounding it. Outside $\Delta$ the $N$-harmonic breather solution does not necessarily vanish and localization of the wavepacket within $\Delta$ only may seem to be very restrictive. Analogy with the linear case, however, tells us that by choosing sufficiently many $\lambda_k$'s within a sufficiently small interval $[\lambda_{\mathrm{Ave}} - \Delta\lambda, \lambda_{\mathrm{Ave}} + \Delta\lambda]$ we can make both $X$ and $T$ as large as we wish, thus creating a wavepacket with a life-span as large as we wish, localized on an interval also as large as we wish. At the moment we cannot prove this analytically, yet formulae (3) have been used to verify it numerically.

Figures 4–7 show time evolution of the superpositions of harmonic breathers with

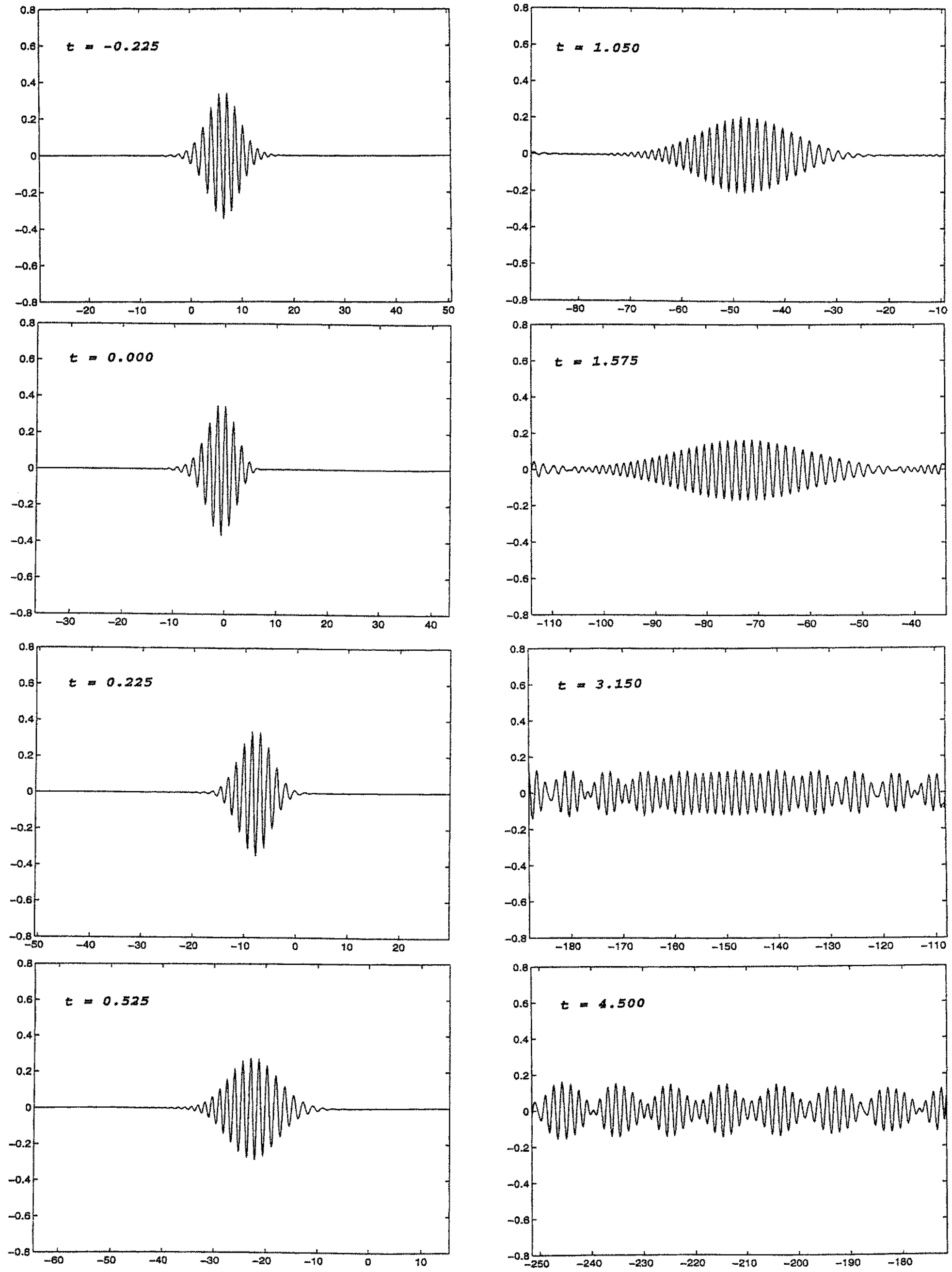


**Figure 3.** (Continued)

uniformly distributed values of $\lambda$, the same value of $\gamma$ and $p_n$ of the form $p_0 n^2$, $p_0 n$ or $p_0$, chosen to mimic the $\delta$-function and its derivatives and antiderivatives. Note that for $t = 0$ figure 6 shows a large negative splash as $x \to 0^-$ and a large positive splash as $x \to 0^+$, whereas figure 7 shows two large negative troughs on both sides of the peak in

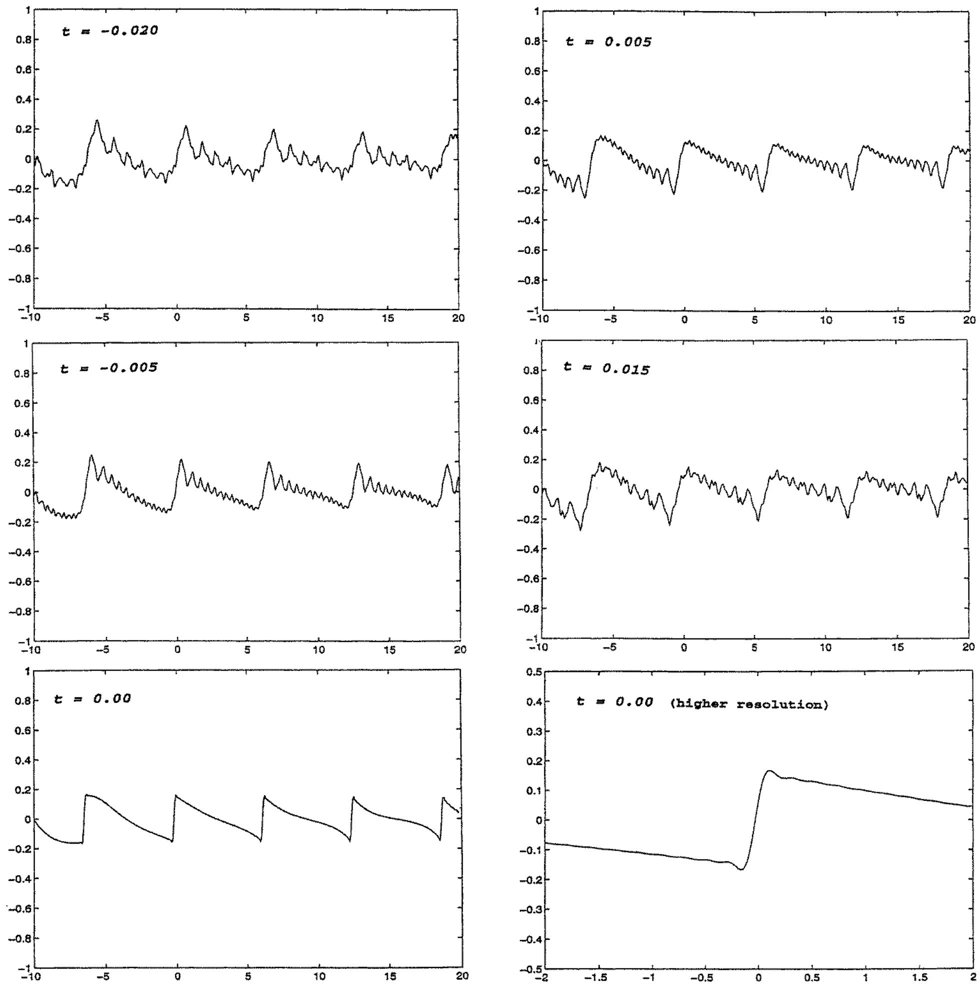


**Figure 4.** Formation and time evolution of the wave generated by 28 harmonic breathers with $\Gamma = (0, 0, \ldots, 0, 0)$; $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{28})$, $\lambda_n = 0.5n$; $P = (p_1, p_2, \ldots, p_{28})$, $p_n = -20n^2$. The value of $t$, corresponding to each frame, is shown in the upper left corner of each frame.

the neighbourhood of $x = 0$. Both the splashes in figure 6 and the troughs in figure 7 are nonlinear analogues of the Gibbs phenomenon.

Of special interest are profiles of the type shown in figure 7, which are soliton-like with a very short life-span. Given the fact that all numerical schemes for (1) are designed for finite intervals only, profiles of figure 7 give an example of initial data that seem to have a soliton in it whereas no soliton is actually present. The difference between the profiles of figure 7 and solitons is actually much deeper than it may seem. Whereas a one-soliton potential has a spectrum that consists of a single point on the imaginary axis, the profiles of figure 7 have no regular spectrum but, instead, have a discrete singular real spectrum [13, 14]. Their behaviour is different for both $x \to \pm\infty$ and $t \to \pm\infty$. Yet for any given but finite interval $|x| \leqslant X$ we can construct an exact solution of KdV like the one in figure 7 whose initial profile is soliton-like on $|x| \leqslant X$. Although we are not aware of any bibliography on the existence of such unstable soliton-like formation for the KdV equation,

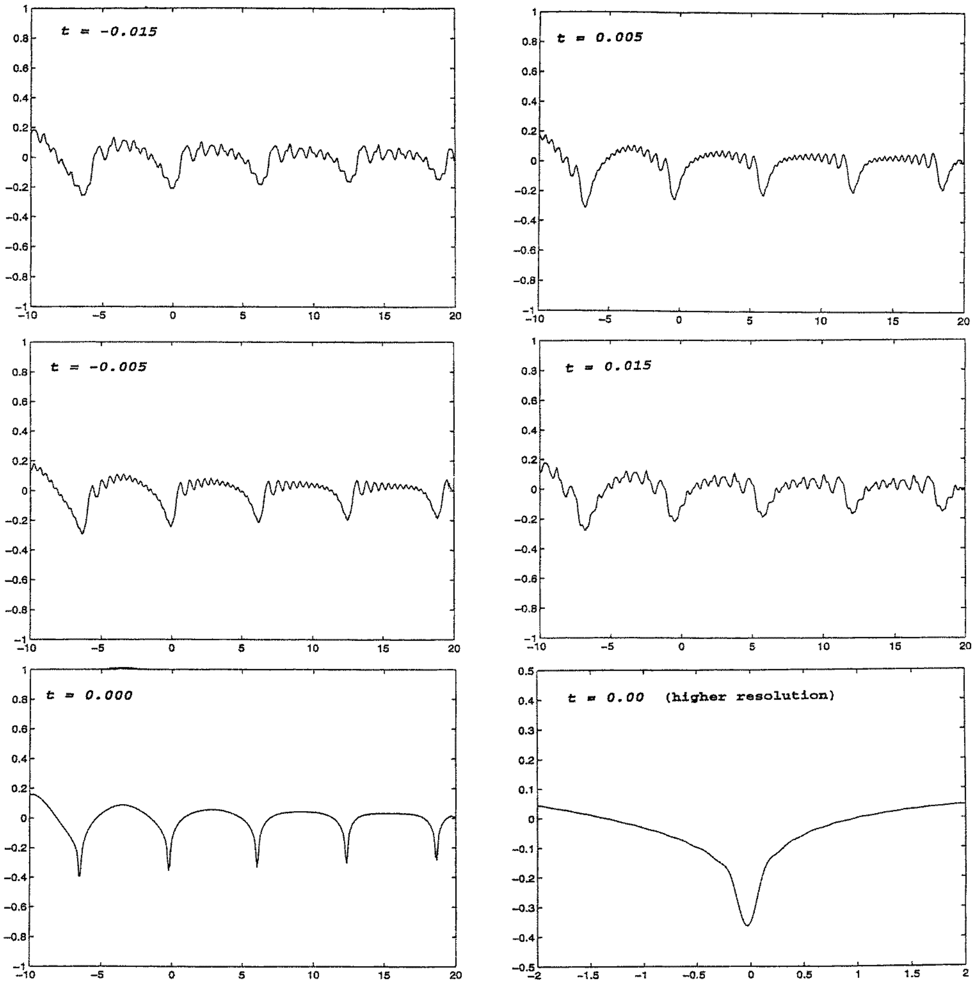


**Figure 5.** Formation and time evolution of the wave generated by 28 harmonic breathers with $\Gamma = (\pi/2, \pi/2, \ldots, \pi/2)$; $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{28})$, $\lambda_n = 0.5n$; $P = (p_1, p_2, \ldots, p_{28})$, $p_n = 20n^2$. The value of $t$, corresponding to each frame, is shown in the upper left corner of each frame.

there is a rather extensive description of such solutions for the generalized KdV, the best reference for this, in our opinion, is [2]. The graphs of [2] clearly indicate that certain solutions of the generalized KdV equation disperse into simpler $e^{i(\lambda x - \omega(\lambda)t)}$-like components similarly to the dispersion taking place for the profiles of figure 7, time evolution of the profiles in [2] and of the profiles of figure 7 are practically identical, and we conjecture that both are essentially of the same nature. Time evolution of the profile of figure 7 is not given in this paper but can be easily obtained using explicit formulae (3).

Using modulation we can also construct 'generalized' solutions of KdV. We illustrate it using the following example. Let $u_{\lambda_{\mathrm{Ave}}, \Delta\lambda, N, P}(x, t)$ denote the superposition of $N$ harmonic breathers with

$$\Lambda = \left(\lambda_n | \lambda_n = \lambda_{\mathrm{Ave}} - \Delta\lambda \left(\frac{N}{2} - n\right)\right)$$

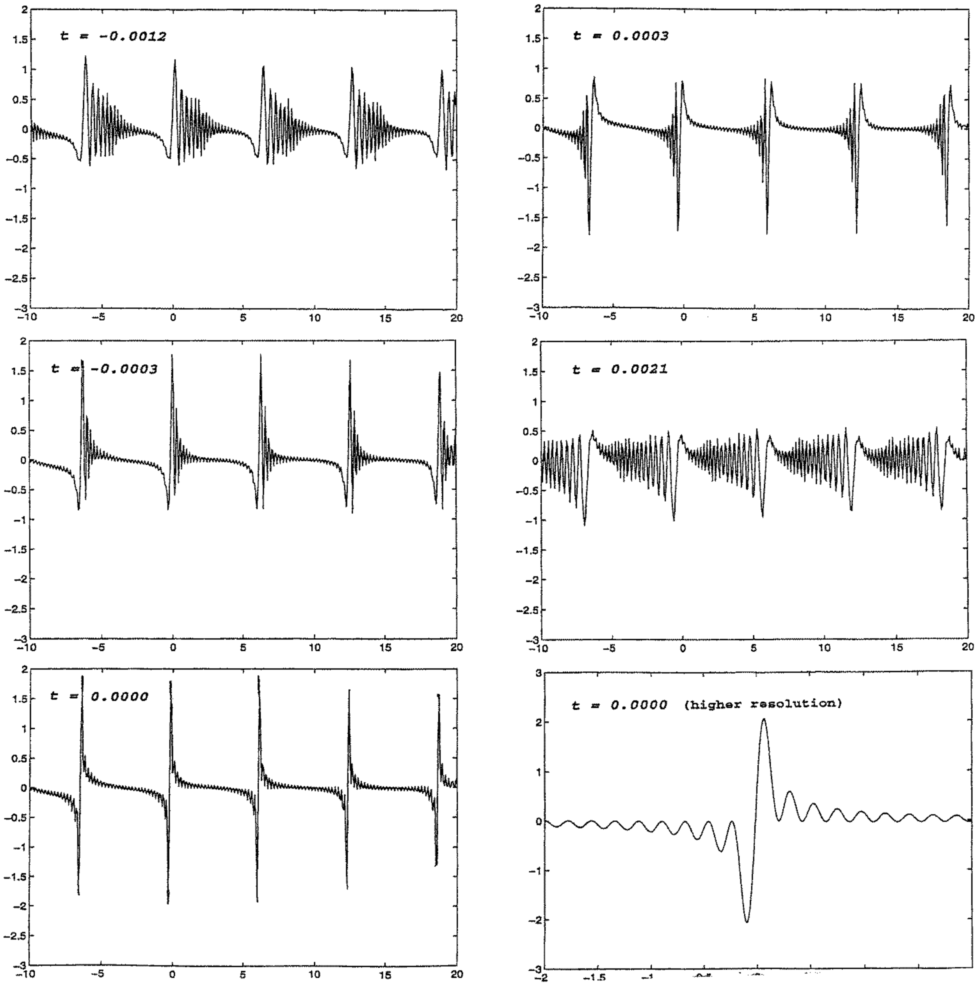


**Figure 6.** Formation and time evolution of the wave generated by 28 harmonic breathers with $\Gamma = (0, 0, \ldots, 0, 0)$; $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{28})$, $\lambda_n = 0.5n$; $P = (p_1, p_2, \ldots, p_{28})$, $p_n = -20n$. The value of $t$, corresponding to each frame, is shown in the upper left corner of each frame.

$$\Gamma = (\gamma_n | \gamma_n = \gamma, 1 \leqslant n \leqslant N) \qquad P = (p_n | p_n = p_n(\Delta\lambda))$$

of the type shown in figure 8. As $N \to +\infty$, $\Delta\lambda \to 0$ and $t = 0$, the sequence $u_{\lambda\mathrm{Ave},\Delta\lambda,N,P}$ converges to something resembling the $\delta$-function, in the sense that its limit is

$$\begin{cases} 0 & x \neq x_0 \\ +\infty & x = x_0 \end{cases}$$

where $x_0$ is a point close to 0. It is interesting to remark that even though we take nonlinear superpositions of harmonic breathers with local maxima at $x = 0$, the peaks of the superpositions in figure 8 do not occur at $x = 0$ but at $x = x_0$, where $x_0$ is a point slightly left of zero. We cannot, however, say that the limit is the $\delta$-function or any other tempered distribution for it has to satisfy (1) but the portion $uu_x$ of (1) is not defined in the sense of distributions. It is only reasonable to say that the sequence determines a certain

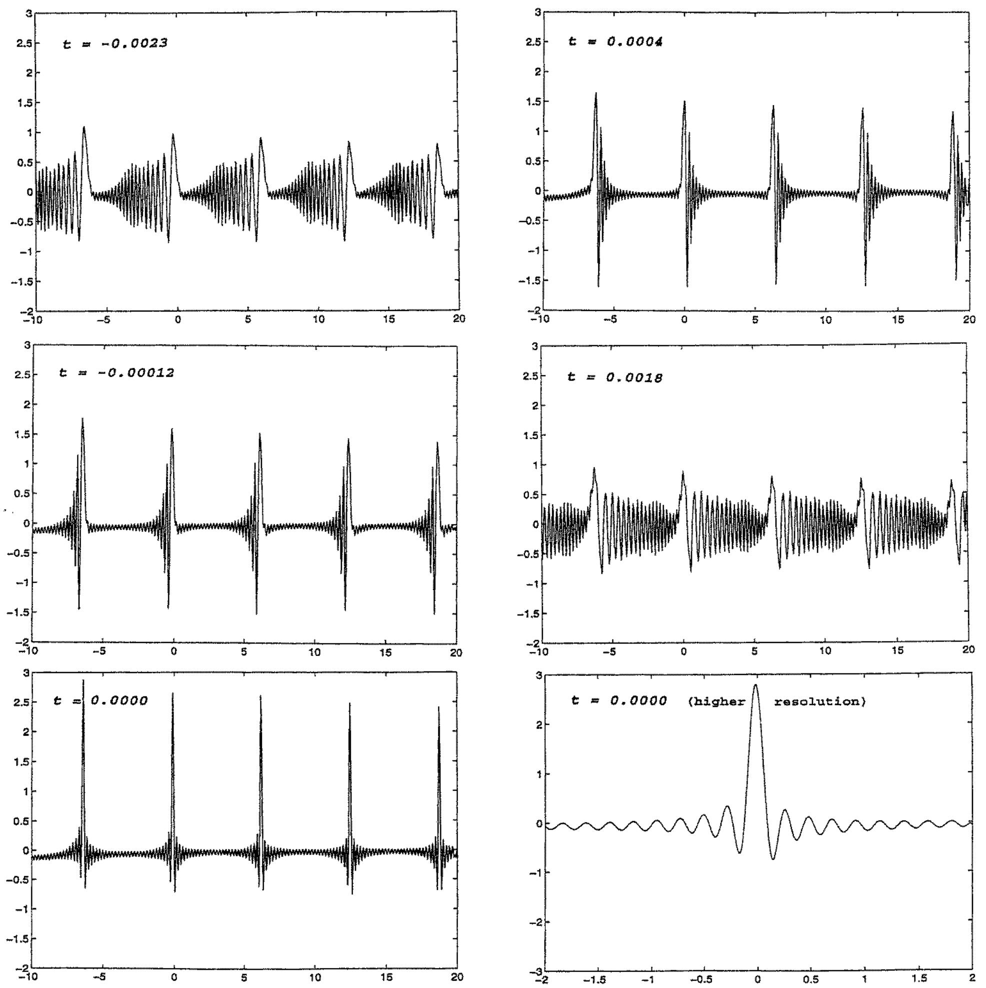


**Figure 7.** Formation and time evolution of the wave generated by 28 harmonic breathers with $\Gamma = (-\pi/2, -\pi/2, \ldots, -\pi/2)$; $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{28})$, $\lambda_n = 0.5n$; $P = (p_1, p_2, \ldots, p_{28})$, $p_n = -20n$. The value of $t$, corresponding to each frame, is shown in the upper left corner of each frame.

new ‘generalized’ solution of KdV whose initial profile is not a generalized function in the classical Sobolev–Schwartz sense. At present there is no theory of such functions although the theory of mnemofunctions discussed in [1] seems to come closest to what we may need.

Although the harmonic breathers were obtained by an appropriate degeneration of the reflection coefficient $r(k)$, we deliberately avoid any further discussion of scattering data and conserved quantities. The reason for that is two-fold. On the one hand $r(k)$ is discontinuous and equation (2) is singular, rendering methods currently used in the inverse scattering theory inapplicable. On the other hand, the sets of parameters $\Lambda = (\lambda_1, \ldots, \lambda_N)$; $\Gamma = (\gamma, \ldots, \gamma)$; $P = (p_1, \ldots, p_N)$are determined by the behaviour of $u(t, x)$ on the interval of modulation rather than on $\mathbb{R}$, again rendering modern methods inapplicable.

The existence of nonlinear modulation, similar in many aspects to its linear namesake, shows that the similarity between the Fourier transform and the inverse scattering theory

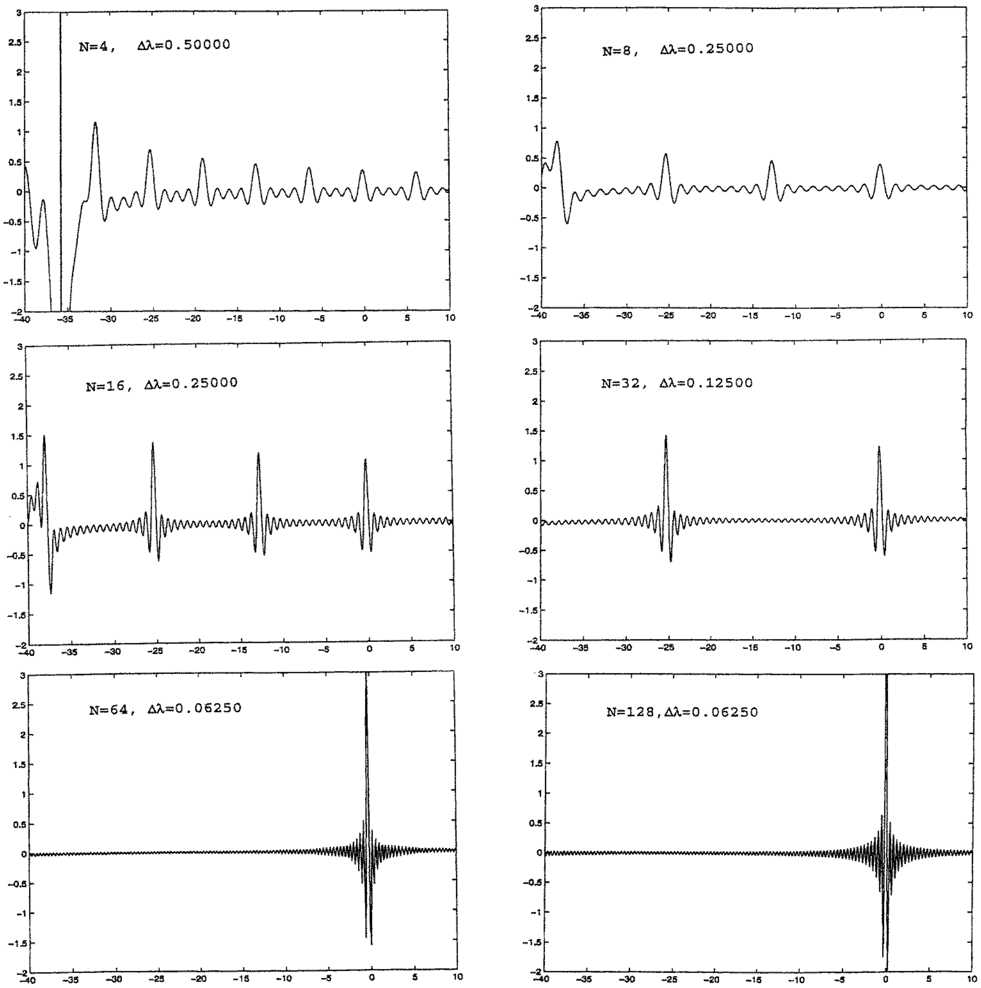


**Figure 8.** Sequence of $N$-harmonic breather solutions converging to a $\delta$-type function. $\Gamma = (-\pi/2, \ldots, -\pi/2)$; $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, $\lambda_n = \Delta\lambda n$, $1 \leqslant n \leqslant N$; $P = (p_1, \ldots, p_N)$, $p_n = -30n^{0.5}\Delta\lambda^{-0.25}$, $t = 0$. The values of $N$ and $\Delta\lambda$ are shown on each frame.

is much deeper than currently believed. Solutions of the KdV equation are dispersive, that is with time they break down into simpler components. In the case of an $N$-soliton solution these components are solitons. In the solitonless case these simpler components have never actually been determined. Moreover, in some applications these components are simply assumed to be approximately $\sin(\lambda x - \omega(\lambda)t)$ and $\cos(\lambda x - \omega(\lambda)t)$ as is the case in derivation of the KdV equation in [20] where the equation is derived under an implicit assumption that its solutions are wavepackets formed by small perturbations of a linear superposition of the trigonometric functions. Yet due to the nonlinearity of the KdV equation the components cannot be trigonometric functions as is the case in the Fourier transform formula. What we have actually shown in this paper is that in at least some cases the simple components are harmonic breathers and with time the wavepackets disperse into them in much the same way as solutions of linear equations disperse into trigonometric functions. This opens up a possibility to provide nonlinear analogues of constructions currently existing in the theory of Fourier transform, for example, nonlinear uncertainty principle, nonlinear creation/annihilation operators, modelling of particles as nonlinear wave/wavepackets, etc [5].

Nonlinear modulation is not particular to the KdV equation only but should also occur for other integrable systems [3, 4], for example, the nonlinear Schrödinger equation for which it actually might be even more important due to its use in nonlinear optics [7, 8, 16, 21].

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## References

[1] Antonevich A B and Radyno Ya V 1991 On a general method for constructing algebras of generalized functions *Sov. Math. Dokl* **43** 3
[2] Bona J L, Karakashian O A, Dougalis V A and McKinney W R 1996 The effect of dissipation on solutions of the generalized Korteweg-de Vries equation *J. Comput. Appl. Math.* **74** 127–54
[3] Beutler R 1992 Positon solutions of the sinh-Gordon equation *MPI für Metallforschung in Stuttgart* Preprint 1-16
[4] Belokolos E D, Bobenko A I, Enolskii V Z, Its A R and Mateev V B 1997 *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Berlin: Springer)
[5] Fuchssteiner B 1996 The interaction equation *Physica* **228A** 189–211
[6] Gardner C S, Greene J M, Kruskal M D and Miura R M 1974 Korteweg–de Vries equation and generalizations VI *Commun. Pure Appl. Math.* **V.XXVII** 97–133
[7] Hasegawa A and Kodama Y 1991 *Phys. Rev. Lett.* **66** 161
[8] Hasegawa A and Tappert F D 1973 *Appl. Phys. Lett.* **23** 142
[9] Hu H 1993 Darboux transformation of Su-chain *Proc Conf. Diff. Geometry in Honor of Professor Su Buchin (Shanghai, 1991)* (Singapore: World Scientific)
[10] Kodama Y and Hasegawa Y 1992 *Opt. Lett.* **17** 31
[11] Kovalyov M 1996 Nonlinear interference and the Korteweg–de Vries equation *Appl. Math. Lett.* **9** 89–92 (Erratum 1997 *Appl. Math. Lett.* **10** 133)
[12] Kovalyov M 1998 Basic motions of the Korteweg–de Vries equation *Nonlinear Anal. Theory Methods Appl.* **31** 599–619
[13] Matveev V B 1994 Asymptotics of the multipositon-soliton $\tau$-function of the Korteweg–de Vries equation and the supertransparency *J. Math. Phys.* **35** 2955
[14] Matveev V B 1992 *Phys. Lett.* A **135** 2009–212
[15] McLaughlin D W 1981 Modulation of KdV wavetrains *Physica* **3D** 335–43
[16] Meccozi A, Moores J D, Haus H A and Lai Y 1991 *Opt. Lett.* **16** 1841
[17] Stahlhofen A A and Maisch H 1995 Dynamical properties of positons *Phys. Scr.* **52** 228
[18] Stahlhofen A A 1995 On completely transparent potentials of the Shrödinger equation *Phys. Rev.* A **51** 934
[19] Stahlhofen A 1992 *Ann. Phys.* **1** 554–69
[20] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[21] Zahkharov V E and Shabat A B 1972 *Sov. Phys.–JETP* **34** 62